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TWO WAYS OF OBTAINING INFINITESIMALS BY REFINING CANTOR'S COMPLETION OF THE REALS

PAOLO GIORDANO AND MIKHAIL G. KATZ

Abstract. Cantor's famous construction of the real continuum in terms of Cauchy sequences of rationals proceeds by imposing a suitable equivalence relation. More generally, the completion of a metric space starts from an analogous equivalence relation among sequences of points of the space. Can Cantor's relation among *Cauchy* sequences of reals be refined so as to produce a Cauchy complete and infinitesimal-enriched continuum? We present two possibilities: one leads to invertible infinitesimals and the hyperreals; the other to nilpotent infinitesimals (e.g. $h \neq 0$ infinitesimal such that $h^2 = 0$) and Fermat reals. One of our themes is the trade-off between formal power and intuition.

Contents

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1. Introduction

In a recent issue of *The American Mathematical Monthly*, K. Hrbacek *et al* have argued that analysis needs better axiomatics than Zermelo-Fraenkel set theory with the axiom of choice (ZFC), see [17]. They propose a new axiomatic framework that naturally includes numbers they call *ultrasmall* (i.e., infinitesimal). They mention [17, p. 803] that these ideas are not entirely new, and provide a list of references the earliest of which is E. Nelson's 1977 text [32], where the author outlined his enrichment of ZFC known as Internal Set Theory (IST).

While the axiomatic approach has much to recommend itself, we feel that a prerequisite for new axiomatics is a good understanding of the mathematical structure that stands to be axiomatized, and not vice versa. To make a convincing case in favor of a new axiom system, one first needs to explain the basics. In the case of infinitesimal-enriched continua, the basics amount to understanding the ultrapower construction. One of our goals in this text is to give an accessible explanation of the latter in the context of Cauchy sequences, as well as providing possible alternatives.

Cantor's completion of the rationals resulting in the field of real numbers proceeds by quotienting the space $\mathcal{C}_{\mathbb{Q}} \subset \mathbb{Q}^{\mathbb{N}}$ of all Cauchy sequences of rational numbers by Cauchy's equivalence relation. Similarly, the collection $\mathcal{C} \subset \mathbb{R}^{\mathbb{N}}$ of all Cauchy sequences of real numbers projects to the Archimedean continuum $\mathbb{R}$:

$$\mathcal{C} \xrightarrow{\lim} \mathbb{R}. \tag{1.1}$$

The corresponding equivalence relation $\sim_{\mathcal{C}}$ defined by

$$u \sim_{\mathcal{C}} v \quad \text{if and only if} \quad \lim_{n\to\infty} |u_n - v_n| = 0,$$

"collapses" all null sequences to a single point $0 \in \mathbb{R}$. Is there another way to define an equivalence relation $\sim$ on the space $\mathcal{C}$ that would allow some null sequences to retain their distinct identity? In other words, can one *refine* Cantor's equivalence relation among Cauchy sequences, in such a way as not to "collapse" all null sequences to zero? The idea would be that, relative to a new equivalence relation $\sim$, a null sequence of reals would become an actual infinitesimal. In other words, we are searching for a new notion of "completion", with respect to which the real field $\mathbb{R}$ can be completed by the addition of infinitesimals. What one seeks is an intermediate stage, ${}^*\mathbb{R}_{\mathrm{f}} := \mathcal{C}/\sim$, in the projection (1.1). The subscript "f" in the symbol ${}^*\mathbb{R}_{\mathrm{f}}$ stands for "finite" (i.e., there are no infinite numbers). Such an intermediate stage ${}^*\mathbb{R}_{\mathrm{f}}$ would represent an infinitesimal-enriched continuum as in Figure 1.1.

$$\begin{tikzcd} & {}^*\mathbb{R}_{\mathrm{f}} \arrow[d, shift left, "\mathrm{st}"] \\ \mathcal{C} \arrow[ur, "p"] \arrow[r, "\lim"'] & \mathbb{R} \arrow[u, hook, shift left] \end{tikzcd}$$

FIGURE 1.1. Factoring Cantor's map $\mathcal{C} \to \mathbb{R}$ through an intermediate stage ${}^*\mathbb{R}_{\mathrm{f}}$.

Here, if $[u]_\sim$ is the new equivalence class of a sequence u, then the function

$$\mathrm{st} : {}^*\mathbb{R}_{\mathrm{f}} \to \mathbb{R},$$

defined by

$$\mathrm{st}([u]_\sim) := \lim_{n\to+\infty} u_n \ \in \mathbb{R}$$

is the usual limit of a Cauchy sequence $u = (u_n)_{n\in\mathbb{N}} \in \mathcal{C}$. This function represents the *standard part* of $[u]_\sim \in {}^*\mathbb{R}_{\mathrm{f}}$, that is a standard real number infinitely close to the new number $[u]_\sim \in {}^*\mathbb{R}_{\mathrm{f}}$. The most natural way to obtain a ring structure on ${}^*\mathbb{R}_{\mathrm{f}}$ is to define the equivalence relation $\sim$ so that it preserves pointwise sums and products. Therefore, we expect ${}^*\mathbb{R}_{\mathrm{f}}$ to be a ring rather than a field, because it cannot contain the pointwise inverse $\left(\frac{1}{u_n}\right)_{n\in\mathbb{N}}$ of an infinitesimal $[u_n]$, since the inverse is not a Cauchy sequence.

In this text, we will explore two possible implementations of these ideas.

2. A possible approach with invertible infinitesimals

To implement the ideas outlined in Section 1, a possible approach is to declare two Cauchy sequences u, $v \in \mathcal{C}$ to be equivalent if they coincide on a "dominant" set of indices in $\mathbb{N}$:

$$u \sim v \quad \iff \quad \{n \in \mathbb{N} \,|\, u_n = v_n\} \text{ is dominant.} \tag{2.1}$$

For simplicity, we will use the symbol $[u]$ for the equivalence class $[u]_\sim$ generated by $u \in \mathcal{C}$.

What is "dominant"? A finite set in $\mathbb{N}$ is never dominant; every cofinite set (i.e., set with finite complement) is necessarily dominant, and we also expect the property that the superset of a dominant set is dominant, as well. Moreover, we expect the relation (2.1) to yield an equivalence relation. In particular, the validity of the transitive

property for generic Cauchy sequences implies that the intersection of two dominant sets is dominant. In fact, let us assume that

$$\forall u, v, w \in \mathcal{C} : \quad u \sim v \ \wedge \ v \sim w \ \Rightarrow \ u \sim w. \tag{2.2}$$

Then, if sets A and B of indices are dominant, it suffices to take[1]

$$u_n := \begin{cases} 1 & \text{if } n \in A \\ 1 - \frac{1}{n+1} & \text{if } n \in \mathbb{N} \setminus A \end{cases} \qquad w_n := \begin{cases} 1 & \text{if } n \in B \\ 1 + \frac{1}{n+1} & \text{if } n \in \mathbb{N} \setminus B \end{cases}$$

to obtain $u \sim 1$ and $1 \sim w$, so that $u \sim w$ from (2.2). It follows that the set $\{n \in \mathbb{N} \mid u_n = w_n\} = A \cap B$ is dominant. Conversely, if our family of dominant sets is closed with respect to finite intersections, then the relation $\sim$ is an equivalence relation. For example, the family of all cofinite sets

$$\mathcal{F} := \{S \subseteq \mathbb{N} \mid \mathbb{N} \setminus S \text{ is finite}\},$$

the so-called Fréchet filter, satisfies all the conditions we have imposed so far on dominant sets. These conditions define the notion of a *filter* on the set $\mathbb{N}$ (extending the Fréchet filter).

It is easy to prove that the equivalence relation $\sim$ preserves pointwise operations

$$[u] + [v] := \left[(u_n + v_n)_{n \in \mathbb{N}}\right] \quad \text{and} \quad [u] \cdot [v] := \left[(u_n \cdot v_n)_{n \in \mathbb{N}}\right] \tag{2.3}$$

so that the quotient

$$^*\mathbb{R}_{\mathrm{f}} := \mathcal{C}/\sim \tag{2.4}$$

becomes a ring. Moreover, the real numbers $\mathbb{R}$ are embedded in $^*\mathbb{R}_{\mathrm{f}}$ as constant sequences. Whether or not $^*\mathbb{R}_{\mathrm{f}}$ is an integral domain depends on the choice of the filter of dominant sets. Thus, the product of sequences u and w given by

$$u_n := \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{n+1} & \text{if } n \text{ is odd} \end{cases} \qquad w_n := \begin{cases} \frac{1}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

is zero, but whether or not $[u]$ is zero depends on whether the set of even numbers is considered to be dominant or not. We will solve this problem later.

To show that the relation as in (2.1) is a refinement of the usual Cauchy relation $\sim_{\mathcal{C}}$, assume that $u, v \in \mathcal{C}$ coincide on a dominant set A. Then we have $u_{\sigma_n} - v_{\sigma_n} = 0$ for some subsequence $\sigma : \mathbb{N} \to \mathbb{N}$ (enumerating the members of the set A). It follows that $u \sim_{\mathcal{C}} v$ since u and v converge. Of course, the relation $\sim$ is a strict refinement because

[1] Let us note that in $\mathbb{N}$ we have $0 \in \mathbb{N}$.

if we take $u_n = \frac{1}{n}$ and $v_n = 0$, then $u \sim_{\mathcal{C}} v$ but $\{n \in \mathbb{N} \mid u_n = v_n\} = \emptyset$ is the empty set, which is never dominant.

Whether the idea expressed by the notion of a dominant set as in (2.1) can be considered "natural" or not is a matter of opinion. An alternative approach would be to define a new equivalence relation in terms of the rate of convergence of the difference $u - v$. A thread going in this direction will be presented in Section 6, but here we will continue with the approach based on (2.1). If one accepts this idea, then it is also natural to define an *order*, by setting

$$[u] \geq [v] \quad \Longleftrightarrow \quad \{n \in \mathbb{N} \mid u_n \geq v_n\} \text{ is dominant.} \tag{2.5}$$

This yields an ordered ring, as one can easily check.

Is this order total? The assumption that it is total, i.e.

$$\forall u \in \mathcal{C}: \ [u] \geq 0 \quad \text{or} \quad [u] \leq 0, \tag{2.6}$$

yields a further condition on dominant sets. In fact, if A is dominant, then defining

$$u_n := \begin{cases} \frac{1}{n+1} & \text{if } n \in A \\ -\frac{1}{n+1} & \text{if } n \in \mathbb{N} \setminus A \end{cases}$$

we have that A is dominant if the first alternative of (2.6) holds; otherwise $\mathbb{N} \setminus A$ is dominant. A filter satisfying this additional condition is called a *free ultrafilter.*[2] Using this additional condition, we are now also able to prove that ${}^*\mathbb{R}_{\mathrm{f}}$ is an integral domain.

Theorem 1. *${}^*\mathbb{R}_f$ is an integral domain.*

Proof. Given nonzero classes $[u] \neq 0$ and $[v] \neq 0$, both of the sets $\{n \in \mathbb{N} \mid u_n \neq 0\}$ and $\{n \in \mathbb{N} \mid v_n \neq 0\}$ are dominant. Therefore so is their intersection. $\square$

Given an integral domain, we can consider the corresponding field of fractions ${}^*\mathbb{R}_{\mathrm{frac}}$. Since ${}^*\mathbb{R}_{\mathrm{f}}$ is also an ordered ring, the order structure extends to the quotient field of fractions in the usual way.

Remark 2. In a classical approach to nonstandard analysis, the equality on a dominant set (Formula (2.1)) is applied to *arbitrary* sequences, rather than merely Cauchy sequences. Nonetheless, our field of fractions ${}^*\mathbb{R}_{\mathrm{frac}}$ is isomorphic to the full hyperreal field[3] ${}^*\mathbb{R}$ of nonstandard

[2] For the sake of completeness, we say that $\mathcal{U} \subseteq \mathcal{P}(I)$ is an ultrafilter on the set I if $\emptyset \notin \mathcal{U}$; $\mathcal{U}$ is closed with respect to finite intersections and supersets; and $X \in \mathcal{U}$ or $I \setminus X \in \mathcal{U}$ for every subset X of I.

[3] Namely, the field obtained as the quotient of $\mathbb{R}^{\mathbb{N}}$ using the same ultrafilter; in general, the result will depend on the ultrafilter.

analysis through

$$\frac{[u]}{[v]} \in {}^*\mathbb{R}_{\text{frac}} \mapsto \left[\left(\frac{u_n}{v_n} \right)_{n\in\mathbb{N}} \right]_{\mathcal{U}} \in {}^*\mathbb{R},$$

where $[(q_n)_n]_{\mathcal{U}}$ is the equivalence class modulo the ultrafilter $\mathcal{U}$. To prove this, note that every sequence $q \in \mathbb{R}^{\mathbb{N}}$ can be written as $q = \frac{u}{v}$ for two null sequences u, v. Thus, we can set $u_n := q_n \cdot \frac{1}{e^{|q_n|}\cdot(n+1)}$ and $v_n := \frac{1}{e^{|q_n|}\cdot(n+1)}$.

3. A free ultrafilter, anyone?

Our intuition yearns for meaningful examples of free ultrafilters. Such can be obtained by using Zorn's lemma. It is possible to prove that some weaker form of the axiom of choice is necessary to prove the existence of a free ultrafilter. The use of this axiom in modern mathematics is routine. Thus, one of its standard consequences is the Hahn-Banach theorem, of fundamental importance in functional analysis.[4] Yet, one consequence of exploiting this axiom is that we don't possess detailed information about how free ultrafilters are made. Moreover, this also implies that it is not so easy to prove the existence of a free ultrafilter satisfying some given and potentially useful conditions.

Theorem 3. *The Fréchet filter can be extended to a free ultrafilter.*

Proof. See Tarski [34] (1930). □

We have to admit that it is not so easy to judge the idea represented by formula (2.1). Indeed, starting from our definition of pointwise operations and order, one can easily guess that this idea is formally very powerful. For example, it is almost trivial to extend to ${}^*\mathbb{R}_{\text{f}}$ the validity of general laws about real numbers, such as the following law:

$$\forall x, y \in \mathbb{R} : \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y). \tag{3.1}$$

In fact, we can extend trigonometric functions pointwise. Namely, we extend $\sin : \mathbb{R} \to \mathbb{R}$ to ${}^*\sin : {}^*\mathbb{R}_{\text{f}} \to {}^*\mathbb{R}_{\text{f}}$ by setting

$${}^*\sin([u]) := \left[(\sin(u_n))_{n\in\mathbb{N}} \right].$$

Finally, the law (3.1) extends to ${}^*\sin$ and ${}^*\cos$ because the set of indices $n \in \mathbb{N}$ where it is true is all of $\mathbb{N}$. We will deal with this extension of laws from $\mathbb{R}$ to ${}^*\mathbb{R}_{\text{f}}$ in more general terms in Section 5.

On the other hand, whatever will be the example of ultrafilter we will be able to present, it doesn't seems sufficiently meaningful why the

[4] Luxemburg [29] explored possibilities of constructing extensions that rely on Hahn-Banach only (rather than full choice).

infinitesimal $\left[\left(\frac{(-1)^n}{n+1}\right)_{n\in\mathbb{N}}\right]$ should be considered positive rather than negative, or vice versa.[5]

To summarize, the idea of requiring sequences to coincide on dominant sets, even if it may seem initially forbidding from an intuitive point of view, appears to be formally extremely powerful. As an alternative, in Section 6 we will present another idea, which is intuitively clear but which doesn't seem equally powerful. Which thread one wishes to follow would depend on applications envisioned.

4. Actual infinitesimals, null sequences, and standard part

What are, formally, the infinitesimals in the ring ${}^*\mathbb{R}_{\mathrm{f}}$ of (2.4), and how are they related to null sequences? An infinitesimal is a number in ${}^*\mathbb{R}_{\mathrm{f}}$ which belongs to every interval of the form $\left[\frac{-1}{n}, \frac{1}{n}\right]$:

Definition 4. We say that $x \in {}^*\mathbb{R}_{\mathrm{f}}$ is *infinitesimal* if and only if

$$\forall n \in \mathbb{N}_{\neq 0} : \ -\frac{1}{n} < x < \frac{1}{n},$$

and we will write $x \approx 0$. Similarly, we write $y \approx z$ if $y - z \approx 0$. Clearly, such an $x \in {}^*\mathbb{R}_{\mathrm{f}}$ will be infinitesimal if and only if in the field of fractions ${}^*\mathbb{R}_{\mathrm{frac}}$, the element x^{-1} is infinite,[6] i.e. it doesn't satisfy a bound of the form $|x| < n$ for some $n \in \mathbb{N}$.

[5] Moreover, examining the conditions defining the notion of ultrafilter, one can guess that the notion of a dominant set is not intuitively so clear. In point of fact, the technically desirable conditions about the closure with respect to intersection and complement can lead to counter intuitive consequences. We would have that even numbers P_2 or odd numbers will be dominant (but not both). Let us suppose, e.g., the first case and continue: even numbers in P_2, i.e. the set P_4 of multiples of 4, or its complement $\mathbb{N} \setminus P_4$ will be dominant. In the latter case, also $P_2 \cap (\mathbb{N} \setminus P_4)$, i.e. numbers of the form $2(2n+1)$, will be dominant. In any case we would be able to find always a dominant set which has "1/2 of the elements of the previous dominant set". Continuing in this way, we can obtain a dominant set, which is intuitively very "thin" with respect to its complement. To understand this idea a little better, let us consider that everything we said up to now can be generalized if instead of sequences $u : \mathbb{N} \to \mathbb{R}$ we take functions $u : [0, 1] \to \mathbb{R}$. In other words, instead of taking our indices as integer numbers, we take real numbers in $[0, 1]$. Then, we can repeat the previous reasoning considering, at each step k, subintervals of length 2^{-k}. Therefore, for every $\varepsilon > 0$, we are always able to find in an ultrafilter on $[0, 1]$ a dominant set A whose uniform probability $P(A) < \epsilon$, whereas $P([0, 1] \setminus A) > 1 - \epsilon$, even though this complement is not dominant. See [8] for a formalization of this idea using the notion of density of subsets of $\mathbb{N}$.

[6] Note that without additional assumptions, ${}^*\mathbb{R}_{\mathrm{frac}}$ may contain additional more infinitesimals not found in ${}^*\mathbb{R}_{\mathrm{f}}$; see Remark 7 below.

Do infinitesimals in ${}^*\mathbb{R}_{\mathrm{f}}$ correspond to ordinary null sequences?

Theorem 5. *Let* $[u] \in {}^*\mathbb{R}_f$, *then we have that*

$$[u] \textit{ is infinitesimal}$$

if and only if

$$\lim_n u_n = 0$$

Proof. Let us assume that $[u]$ is infinitesimal, then for each $n \in \mathbb{N}_{\neq 0}$, the set

$$A_n := \left\{ k \in \mathbb{N} \,:\, -\frac{1}{n} < u_k < \frac{1}{n} \right\}$$

is dominant. Therefore, it is infinite and we can always find an increasing sequence $k : \mathbb{N} \to \mathbb{N}$ such that $k_n \in A_n$ and $k_{n+1} > k_n$. For such a sequence we have

$$\forall n \in \mathbb{N}_{\neq 0} : \ -\frac{1}{n} < u_{k_n} < \frac{1}{n}.$$

Since $u \in \mathcal{C}$ is a Cauchy sequence, we obtain

$$\lim_{n \to +\infty} u_n = \lim_{n \to +\infty} u_{k_n} = 0.$$

To prove the converse implication, we can consider that

$$\forall n \in \mathbb{N}_{\neq 0} \, \exists N : \ \forall k \in \mathbb{N}_{\geq N} : \frac{1}{n} < u_k < \frac{1}{n}.$$

Since every cofinite set $\mathbb{N}_{\geq N}$ is dominant, this proves that $[u]$ is infinitesimal. $\square$

As a corollary, we have that $[u] \approx [v]$ if and only if $\lim_n u_n = \lim_n v_n$. This allows us to define the standard part mentioned above.

Definition 6. Let $[u] \in {}^*\mathbb{R}_{\mathrm{f}}$, then the real number

$$\mathrm{st}([u]) := \lim_{n \to +\infty} u_n \in \mathbb{R} \tag{4.1}$$

is called the *standard part* of $[u]$.

Note that we have $x \approx \mathrm{st}(x)$ for every $x \in {}^*\mathbb{R}_{\mathrm{f}}$.

Is our extension ${}^*\mathbb{R}_{\mathrm{f}}$ of $\mathbb{R}$ still Cauchy complete with respect to some kind of metric extending the usual Euclidean metric on $\mathbb{R}$? It is not hard to prove that

$$d\,(x, y) := |\mathrm{st}(x) - \mathrm{st}(y)| \in \mathbb{R} \quad \forall x, y \in {}^*\mathbb{R}_{\mathrm{f}}$$

defines a pseudo-metric having the desired properties. Note that ${}^*\mathbb{R}_{\mathrm{f}}$ is not Dedekind complete, since the set of all the infinitesimals is bounded but does not admit a least upper bound.

Remark 7. In the quotient field ${}^*\mathbb{R}_{\text{frac}}$, the assertion of Theorem 5 is not generally true, unless one considers a particular type of ultrafilter, called a P-point. While the existence of a free ultrafilter can be proved using Zorn's lemma, which is equivalent to the axiom of choice, the existence of a P-point cannot be proved in ZFC, that is using the usual axioms of set theory plus the axiom of choice. Assuming the continuum hypothesis or Martin's axiom and using transfinite induction, it is possible to prove the existence of a P-point. See [5] and references therein for more details about this foundational wrinkle.[7]

5. Leibniz's law of continuity in ${}^*\mathbb{R}_{\text{F}}$

To convey the full power of the idea (2.1), we have to go back to Leibniz. Leibniz introduced infinitesimal and infinite quantities, and developed a heuristic principle called the "law of continuity", which had roots in the work of earlier scholars such as Nicholas of Cusa and Johannes Kepler. It is the principle that:

> What succeeds for the finite numbers succeeds also for the infinite numbers

(see Knobloch [25, p. 67], Robinson [33, p. 266], and Laugwitz [27]).

Kepler had already used it to calculate the area of the circle by representing the latter as an infinite-sided polygon with infinitesimal sides, and summing the areas of infinitely many triangles with infinitesimal bases. Leibniz used the law to extend concepts such as arithmetic operations, from ordinary numbers to infinitesimals, laying the groundwork for infinitesimal calculus.

Of course, a modern mathematical version of this heuristic law depends on our formalization of the first word 'what' in the law of continuity as stated above. We have already seen that this is almost trivial if what we mean by the Leibnizian 'what' is "continuous equalities between real numbers". In fact we can extend arbitrary continuous functions as follows. Recall that $\mathcal{C}$ is the space of Cauchy sequences of real numbers.

[7] The previous Theorem 5 can be easily extended to ${}^*\mathbb{R}_{\text{frac}}$ if we consider fractions $\frac{[u]}{[v]}$ for which the limit $\lim_{n\to+\infty} \frac{u_n}{v_n}$ exists finite. It results that $\frac{[u]}{[v]}$ is infinitesimal in ${}^*\mathbb{R}_{\text{frac}}$ if and only the limit of this fraction is zero. This permits to extend the definition of the standard part function to all the fractions which are of the form $\frac{0}{0}$ but whose limit exists finite. Finally, because of the previous Remark 7 and of the isomorphism ${}^*\mathbb{R}_{\text{frac}} \simeq {}^*\mathbb{R}$, we can always find an infinitesimal $\frac{[u]}{[v]} \in {}^*\mathbb{R}_{\text{frac}}$ which is not generated by an infinitesimal sequence $\left(\frac{u_n}{v_n}\right)_n$; of course, this fraction is of the form $\frac{0}{0}$ but the corresponding ratio of sequences doesn't converge.

Definition 8. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Then we have $f \circ (u^1, \dots, u^d) \in \mathcal{C}$ for every d-tuple of Cauchy sequences $u^1, \dots, u^d \in \mathcal{C}$, and we can define the extension ${}^*f$ by setting

$$^*f\left([u^1], \dots, [u^d]\right) := \left[\left(f(u^1_n, \dots, u^d_n)\right)_{n\in\mathbb{N}}\right]_\sim \quad \forall [u^1], \dots, [u^d] \in {}^*\mathbb{R}_{\mathrm{f}}.$$

This gives a true extension of f, i.e. ${}^*f(r_1, \dots, r_d) = f(r_1, \dots, r_d)$ for every $r_1, \dots, r_d \in \mathbb{R}$ (identified with the corresponding constant sequences).

Theorem 9. *Let f, $g : \mathbb{R}^d \to \mathbb{R}$ be continuous functions, then the equality*

$$\forall x_1, \dots, x_d \in \mathbb{R} : \; f(x_1, \dots, x_d) = g(x_1, \dots, x_d) \tag{5.1}$$

is satisfied if and only if

$$\forall \alpha_1, \dots, \alpha_d \in {}^*\mathbb{R}_f : \; {}^*f(\alpha_1, \dots, \alpha_d) = {}^*g(\alpha_1, \dots, \alpha_d) \tag{5.2}$$

Analogously, we can formulate the transfer of inequalities of the form $f(x_1, \dots, x_d) < g(x_1, \dots, x_d)$.

Proof. The equality (5.1) implies

$$\left\{n \in \mathbb{N} \mid f(a^1_n, \dots, a^d_n) = g(a^1_n, \dots, a^d_n)\right\} = \mathbb{N},$$

where $[a^k] = \alpha_k$. The whole set $\mathbb{N}$ is dominant, and therefore (5.2) follows. The converse implication follows from the fact that ${}^*f$ and ${}^*g$ extend f and g and from the embedding $\mathbb{R} \subset {}^*\mathbb{R}_{\mathrm{f}}$. □

Remark 10. The reader would have surely noted that some of the limitations we have presented can be avoided by generalizing our construction further. For example, the ring ${}^*\mathbb{R}_{\mathrm{f}}$ is only an integral domain and not a field, because it is not closed with respect to pointwise inverse, because the latter are not Cauchy sequences. Similarly, we cannot extend a general function $f : \mathbb{R}^d \to \mathbb{R}$ but only continuous functions because we need to ensure that the image sequence is Cauchy. However, all the ideas we have introduced up to now work if we replace $\mathcal{C}$ by the whole of $\mathbb{R}^{\mathbb{N}}$. In this way, we obtain a field ${}^*\mathbb{R}$ and arbitrary functions can be extended. In the setting of the ring ${}^*\mathbb{R}_{\mathrm{f}}$ only continuous functions can be extended and hence a continuity hypothesis has to be assumed if one wants to use its infinitesimals. For more details, see Goldblatt [14]. In the present article, we adhere to the framework of defining a new notion of completeness so as to add new infinitesimal points to $\mathbb{R}$. We will motivate our definition by developing some powerful key properties of an infinitesimal-enriched extension of $\mathbb{R}$.

Can Leibniz's law of continuity be proved for more general properties, e.g. for order relations or disjunctions of equality and inequality or even more general relations? To solve this problem, we start, once again, from a historical consideration.

Cauchy used infinitesimals to define continuity as follows: a function f is continuous between two bounds if for all x between those bounds, the difference $f(x+h)-f(x)$ will be infinitesimal whenever h is infinitesimal, see [4].

Such a definition tends to bewilder a modern reader, used to thinking of f as being defined for real values of the variable x, but now we can think of $f(x+h)$ as corresponding to ${}^*f(x+h)$. The function f is not necessarily defined on all of $\mathbb{R}$, so that an extension of the real domain D of the function is implicit in Cauchy's construction. Therefore, we will start by defining such an extension of $D \subseteq \mathbb{R}$. We will first define the symbol "$\in_n$", and then define ${}^*D_{\mathrm{f}}$ in terms of $\in_n$.

Definition 11. Let $u \in \mathcal{C}$ be a Cauchy sequence and $D \subseteq \mathbb{R}$, then

(1) $u_n \in_n D \quad \Longleftrightarrow \quad \{n \in \mathbb{N} \mid u_n \in D\}$ is dominant
(2) ${}^*D_{\mathrm{f}} := \{[u] \in {}^*\mathbb{R}_{\mathrm{f}} \mid u_n \in_n D\}$

Let us note that the variable n is mute in the notation $u_n \in_n D$.

Using this notation, our questions concerning Leibniz's law of continuity can be formulated as preservation properties of the operator ${}^*(-)_{\mathrm{f}}$. In fact, as in Theorem 9, where equalities between continuous functions are preserved, we can ask whether ${}^*(-)_{\mathrm{f}}$ preserves intersections (i.e. "and"), unions (i.e. "or"), set-theoretic difference (i.e. "not"), inclusions (i.e. "if... then..."), etc. To this end, it is interesting to note that a minimal set of extension properties necessary implies ultrafilter conditions.

We will use a circle superscript $^\circ$ in place of a star to indicate a general extension.

Theorem 12. *Assume that* $^\circ(-) : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(^\circ\mathbb{R})$ *preserves unions, intersections and complements, i,.e. for every* A, $B \subseteq \mathbb{R}$, *we have*

$$^\circ(A \cup B) = {}^\circ A \cup {}^\circ B$$
$$^\circ(A \cap B) = {}^\circ A \cap {}^\circ B$$
$$^\circ(A \setminus B) = {}^\circ A \setminus {}^\circ B.$$

Finally, let $e \in {}^\circ\mathbb{R}$. *Then*

$$\mathcal{R}_e := \{X \subseteq \mathbb{R} \mid e \in {}^\circ X\} \text{ is an ultrafilter on } \mathbb{R},$$

and if $e \in {}^\circ\mathbb{N}$, *then*

$$\mathcal{N}_e := \{X \cap \mathbb{N} \mid X \in \mathcal{R}_e\} \text{ is an ultrafilter on } \mathbb{N}.$$

Proof. We need first to prove that $^\circ(-)$ preserves also the empty set and inclusions. Indeed, $^\circ\emptyset = {^\circ}(\emptyset \setminus \emptyset) = {^\circ}\emptyset \setminus {^\circ}\emptyset = \emptyset$. Assume $A \subseteq B$, so that $A = A \cap B$ and $^\circ A = {^\circ}A \cap {^\circ}B$ and thus $^\circ A \subseteq {^\circ}B$.

If $X, Y \in \mathcal{R}_e$, then $e \in {^\circ}X \cap {^\circ}Y = {^\circ}(X \cap Y)$, and hence $X \cap Y \in \mathcal{R}_e$. If $X \in \mathcal{U}_e$ and $\mathbb{R} \supseteq Y \supseteq X$, then $e \in {^\circ}X \subseteq {^\circ}Y$ and hence $Y \in \mathcal{R}_e$. If $X \subseteq \mathbb{R}$, then $\mathbb{R} = X \cup (\mathbb{R} \setminus X)$; but $e \in {^\circ}\mathbb{R} = {^\circ}X \cup ({^\circ}\mathbb{R} \setminus {^\circ}X)$ and therefore $X \in \mathcal{R}_e$ or $\mathbb{R} \setminus X \in \mathcal{R}_e$, and this finally proves that $\mathcal{R}_e$ is an ultrafilter on $\mathbb{R}$ because every $X \in \mathcal{R}_e$ is not empty since $^\circ(-)$ preserves the empty set.

The proof that $\mathcal{N}_e$ is closed with respect to intersection is direct. Consider $\mathbb{N} \supseteq S \supseteq X \cap \mathbb{N}$ with $X \in \mathcal{R}_e$; then $Y := (S \setminus X) \cup X \supseteq X$ and hence $Y \in \mathcal{R}_e$. Therefore, $Y \cap \mathbb{N} = S$ because $X \cap \mathbb{N} \subseteq S$, and hence $S \in \mathcal{N}_e$. Finally, if $S \subseteq \mathbb{N}$, then either $S \in \mathcal{R}_e$, and thus $S = S \cap \mathbb{N} \in \mathcal{N}_e$, or $\mathbb{R} \setminus S \in \mathcal{R}_e$. In the second case, $(\mathbb{R} \setminus S) \cap \mathbb{N} = \mathbb{N} \setminus S \in \mathcal{N}_e$. Up to now, we didn't need the further hypothesis $e \in {^\circ}\mathbb{N}$. However, in this case, if $X \in \mathcal{R}_e$, then $e \in {^\circ}X \cap {^\circ}\mathbb{N} = {^\circ}(X \cap \mathbb{N}) \neq \emptyset$ and hence also $X \cap \mathbb{N} \neq \emptyset$. □

Taking, e.g., $e = 1 \in {^*}\mathbb{N}_{\mathrm{f}}$, yields an ultrafilter (a so-called principal ultrafilter, see the next Corollary 13).

The meaning of this theorem is the following: if one doesn't like the idea (2.1) but wants to obtain something corresponding to Leibniz's law of continuity, one must face the problem that the corresponding extension operator $^\circ(-)$ cannot preserves "and", "or" and "not" of arbitrary subsets. In Section 6, where we will introduce another idea to refine Cauchy's equivalence relation without using ultrafilters, we will see that a corresponding law of continuity holds, but only for open subsets, so that we are forced to define a set-theoretical difference with values in open sets

$$A \setminus B := \mathrm{int}(A \setminus B),$$

where $\mathrm{int}(-)$ is the interior operator. Note that the use of open sets and this "not" operator correspond to the semantics of intuitionistic logic.

For the sake of completeness, we also add the following results, which represents particular cases of the previous Theorem 12.

Corollary 13. *In the hypotheses of Theorem 12, if*

$$X \subseteq {^\circ}X \ , \ ({^\circ}X \setminus X) \cap \mathbb{R} = \emptyset \quad \forall X \subseteq \mathbb{R}, \tag{5.3}$$

then we have that $e \in \mathbb{R}$ if and only if $\mathcal{R}_e$ is the principal ultrafilter generated by *e, i.e.*

$$\mathcal{R}_e = \{X \subseteq \mathbb{R} \,|\, e \in X\}\,. \tag{5.4}$$

Proof. Let us assume that $e \in \mathbb{R}$ and prove the equality (5.4). If $e \in X \subseteq \mathbb{R}$, then $e \in {}^{\circ}X$ because $X \subseteq {}^{\circ}X$ by hypotheses, and therefore $X \in \mathcal{R}_e$. Vice versa if $e \in {}^{\circ}X$, then ${}^{\circ}X = X \cup ({}^{\circ}X \setminus X)$ and hence $e \in X$ because, by hypotheses, $({}^{\circ}X \setminus X) \cap \mathbb{R} = \emptyset$ and $e \in \mathbb{R}$.

Finally, the converse implication follows directly from the equality (5.4) and from $\mathbb{R} \in \mathcal{R}_e$. □

Therefore, if the extension operation ${}^{\circ}X$ really extends X (first condition of (5.3)) adding only new non real points (second condition of (5.3)), then taking $e \in \mathbb{R}$ we get a trivial ultrafilter. However, in our construction we started from a free ultrafilter; this is the case considered in the following corollary.

Corollary 14. *In the hypotheses of Corollary 13, let us assume that* $({}^{\circ}\mathbb{R}, \leq)$ *is an ordered set extending the usual order relation on the reals. Suppose that* $e \in {}^{\circ}\mathbb{R} \setminus \mathbb{R}$ *is* infinite *with respect to* $({}^{\circ}\mathbb{R}, \leq)$*, i.e.*

$$\forall N \in \mathbb{N} : \ e > N$$

and also that

$$\forall N \in \mathbb{N} : \ e > N \ \Rightarrow \ e \in {}^{\circ}[N, +\infty),$$

then the ultrafilter $\mathcal{N}_e$ *is free.*

For example, the field ${}^*\mathbb{R}_{\text{frac}}$ satisfies the hypotheses of this corollary if we take $e = \frac{[1]}{\left[\left(\frac{1}{n}\right)_n\right]}$.

Proof. By our hypothesis, every interval $[N, +\infty) = \{x \in \mathbb{R} \mid x \geq N\}$ is in $\mathcal{R}_e$, therefore $[N, +\infty) \cap \mathbb{N} \in \mathcal{N}_e$. If $X \subseteq \mathbb{N}$ is cofinite, then $\mathbb{N} \setminus X \subseteq [0, N)$ for some $N \in \mathbb{N}$ and hence $X \supseteq [N, +\infty) \cap \mathbb{N}$. From Theorem 12, we have that $\mathcal{N}_e$ is an ultrafilter, so that it is closed with respect to supersets, and hence $X \in \mathcal{N}_e$. □

Our operator ${}^*(-)_{\text{f}}$ has the following preservation properties of propositional logic operators.

Theorem 15. *Let* A*,* $B \subseteq \mathbb{R}$*, then the following preservation properties hold*

(1) ${}^*(A \cup B)_{\text{f}} = {}^*A_{\text{f}} \cup {}^*B_{\text{f}}$
(2) ${}^*(A \cap B)_{\text{f}} = {}^*A_{\text{f}} \cap {}^*B_{\text{f}}$
(3) ${}^*(A \setminus B)_{\text{f}} = {}^*A_{\text{f}} \setminus {}^*B_{\text{f}}$
(4) $A \subseteq B$ if and only if ${}^*A_{\text{f}} \subseteq {}^*B_{\text{f}}$
(5) ${}^*\emptyset_{\text{f}} = \emptyset$
(6) ${}^*A_{\text{f}} = {}^*B_{\text{f}}$ if and only if $A = B$.

Proof. For example, we will prove the preservation of unions, the other proofs being similar. Take $[u] \in {}^*(A \cup B)_{\mathrm{f}}$, then $\{n \,|\, u_n \in A \cup B\}$ is dominant. If $\{n \,|\, u_n \in A\}$ is dominant, then $[u] \in {}^*A_{\mathrm{f}}$; vice versa, $\{n \,|\, u_n \notin A\}$ is dominant and therefore it is also the intersection

$$\{n \,|\, u_n \in A \cup B\} \cap \{n \,|\, u_n \notin A\} = \{n \,|\, u_n \in B\}\,,$$

so that $[u] \in {}^*B_{\mathrm{f}}$. Vice versa, if e.g. $[u] \in {}^*A_{\mathrm{f}}$, then $\{n \,|\, u_n \in A\}$ is dominant, and hence also the superset $\{n \,|\, u_n \in A \cup B\}$ is dominant, i.e. $[u] \in {}^*(A \cup B)_{\mathrm{f}}$. □

Example 16. Let A, B, $C \subseteq \mathbb{R}$ and write e.g. $A(x)$ to mean $x \in A$. We want to see that our previous Theorem 15 implies that Leibniz's law of continuity applies to complicated formulas like

$$\forall x \in \mathbb{R} : \; A(x) \;\Rightarrow\; [B(x) \text{ and } (C(x) \;\Rightarrow\; D(x))]\,. \tag{5.5}$$

In other words, we will show how to apply the previous theorem to show that (5.5) holds if and only if the following formula holds

$$\forall x \in {}^*\mathbb{R}_{\mathrm{f}} : \; {}^*A_{\mathrm{f}}(x) \;\Rightarrow\; [{}^*B_{\mathrm{f}}(x) \text{ and } ({}^*C_{\mathrm{f}}(x) \;\Rightarrow\; {}^*D_{\mathrm{f}}(x))]\,, \tag{5.6}$$

where e.g. ${}^*A_{\mathrm{f}}(x)$ means $x \in {}^*A_{\mathrm{f}}$. In fact, if we assume (5.5), this implies that $A \subseteq B$ and hence, by Theorem 15, ${}^*A_{\mathrm{f}} \subseteq {}^*B_{\mathrm{f}}$. Therefore, if we assume ${}^*A_{\mathrm{f}}(x)$, for $x \in {}^*\mathbb{R}_{\mathrm{f}}$, from this we immediately obtain ${}^*B_{\mathrm{f}}(x)$. The hypotheses (5.5) also implies that $A \cap C \subseteq D$, so that if we further assume ${}^*C_{\mathrm{f}}(x)$ we also obtain that ${}^*D_{\mathrm{f}}(x)$ holds, and this concludes the proof of (5.6). Analogously we can prove the opposite implication.

Remark 17. Of course, the previous example can be generalized to every logical formula, proceeding by induction on the length of the formula, but this requires the usual (simple) background of (elementary) formal logic. As it is well known (see e.g. [30, 15, 2, 24]), our example further shows that this "more advanced" use of nonstandard analysis can be left only to selected readers.

Now, the next problem is natural: what about the preservation of existential and universal quantifier? We have already considered the case of logical connectives like "and", "or", "not" without stressing too much the need to have a background in formal logic. This permits to simplify our presentation and opens this type of setting to a more general audience, including physicists and engineers. We wish to retain the same attitude also toward quantifiers. For this goal we consider two sets X, $Y \subseteq \mathbb{R}$ and the projection $p_X : X \times Y \to X$, $p_X(x, y) = x$, and $C \subseteq X \times Y$, i.e. a relation of the form $C(x, y)$ with $x \in X$ and $y \in Y$.

We have

$$\begin{aligned} p_X(C) &= \{x \in X \mid \exists z \in C :\ x = p_X(z)\} = \\ &= \{x \in X \mid \exists y \in Y :\ C(x,y)\}\,; \\ X \setminus p_X\left[(X \times Y) \setminus C\right] &= \{x \in X \mid \neg\left(\exists y \in Y :\ (x,y) \notin C\right)\} = \\ &= \{x \in X \mid \forall y \in Y :\ C(x,y)\}\,. \end{aligned}$$

Therefore, now our aim is to prove that ${}^*(-)_{\mathrm{f}}$ preserves $p_X(C)$, which corresponds to the existential quantifier (preservation of universal quantifier follows from this and from the preservation of difference). Only here we notice that, exactly as we proceeded for functions considering only the continuous ones, we need an analogous condition for a relation: what is a *continuous relation* $C \subseteq X \times Y$? To find the corresponding definition, we start from the idea that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then we expect that the relation $\{(x,y) \in X \times Y \mid y = f(x)\}$ is continuous. We can therefore note that the peculiarity of the definition of the extension ${}^*f$ (see Definition 8) is that continuity permits us to define ${}^*f$ on all of ${}^*\mathbb{R}_{\mathrm{f}}$. Otherwise, we would always have the possibility to define ${}^*f$ on the smaller domain

$$\{[u] \in {}^*\mathbb{R}_{\mathrm{f}} \mid f \circ u \in \mathcal{C}\}\,.$$

For this reason, we start by introducing the following definition.

Definition 18. Let $X, Y \subseteq \mathbb{R}$ and $C \subseteq X \times Y$, then

$${}^*C_{\mathrm{f}} := \{([u],[v]) \in {}^*X_{\mathrm{f}} \times {}^*Y_{\mathrm{f}} \mid (u_n, v_n) \in_n C\}\,.$$

Next, we compare $\mathrm{dom}({}^*C_{\mathrm{f}})$ and ${}^*[\mathrm{dom}(C)]_{\mathrm{f}}$ as follows.

Theorem 19. *In the previous hypothesis, we always have*

$$\mathrm{dom}({}^*C_{\mathrm{f}}) \subseteq {}^*[\mathrm{dom}(C)]_{\mathrm{f}}$$

$$\mathrm{cod}({}^*C_{\mathrm{f}}) \subseteq {}^*[\mathrm{cod}(C)]_{\mathrm{f}},$$

where $\mathrm{dom}(C) = \{x \in X \mid \exists y \in Y :\ C(x,y)\}$ *is the domain of* C*, and* $\mathrm{cod}(C) = \{y \in Y \mid \exists x \in X :\ C(x,y)\}$ *is the codomain of* C*.*

Proof. We prove, e.g., the relation about the domains. If $[u] \in \mathrm{dom}({}^*C_{\mathrm{f}})$, then there exists v such that $([u],[v]) \in {}^*C_{\mathrm{f}}$, i.e. $u_n \in_n \mathrm{dom}(C)$, and this means that $[u] \in {}^*[\mathrm{dom}(C)]_{\mathrm{f}}$. □

Therefore, it is the opposite inclusion that represents our idea of a continuous relation.

Definition 20. In the previous hypothesis, we say that:

(1) C is continuous in the domain iff $\mathrm{dom}({}^*C_{\mathrm{f}}) \supseteq {}^*[\mathrm{dom}(C)]_{\mathrm{f}}$.
(2) C is continuous in the codomain iff $\mathrm{cod}({}^*C_{\mathrm{f}}) \supseteq {}^*[\mathrm{cod}(C)]_{\mathrm{f}}$.

For example, in the case $C = \mathrm{graph}(f)$, the continuity in the domain says that ${}^*f$ is defined on the whole ${}^*X_{\mathrm{f}}$. Analogously, we can define the continuity of an n-ary relation with respect to its k-th slot.

Theorem 21. *If X, $Y \subseteq \mathbb{R}$, and $f : X \to Y$, then f is continuous if and only if graph(f) is continuous in the domain.*

The proof of this theorem can be directly deduced from the following consideration. The continuity of C in the domain can be written as

$$\forall u \in \mathcal{C} : \ u_n \in_n \mathrm{dom}(C) \quad \Rightarrow \quad \exists y \in {}^*Y_{\mathrm{f}} : {}^*C_{\mathrm{f}}\left([u], y\right). \tag{5.7}$$

We can write this condition in a more meaningful way if we use the following notation for an arbitrary property $\mathcal{P}(n)$:

$$\left[\forall^{\mathrm{d}} n : \ \mathcal{P}(n)\right] \quad \Longleftrightarrow \quad \{n \in \mathbb{N} \,|\, \mathcal{P}(n)\} \text{ is dominant.}$$

For example, $u_n \in_n D$ can now be written as $\forall^{\mathrm{d}} n : \ u_n \in D$. Therefore, (5.7) can be written as

$$\forall u \in \mathcal{C} : \left(\forall^{\mathrm{d}} n \, \exists y \in Y : \ C(u_n, y)\right) \ \Rightarrow \ \exists y \in {}^*Y_{\mathrm{f}} : \ {}^*C_{\mathrm{f}}([u], y). \tag{5.8}$$

This can be meaningfully interpreted in the following way: if we are able to solve the equation

$$C(u_n, y_n) = \mathrm{true}$$

finding a solution $y_n \in Y$ for a dominant set of indices n, then we are also able to solve the equation

$${}^*C_{\mathrm{f}}([u], y) = \mathrm{true}$$

for a solution $y \in {}^*Y_{\mathrm{f}}$.

Using this formulation, it is not hard to prove that all the relations $=$, $<$ and $\leq$ are continuous both in the domain and in the codomain. An expected example of non continuous relation is $x \cdot y = 1$ (take, e.g., $u_n := \frac{1}{n+1}$ in (5.8)). This corresponds to the non applicability of Leibniz's law of continuity to the field property

$$\forall x \in \mathbb{R} : \ x \neq 0 \ \Rightarrow \ \exists y \in \mathbb{R} : \ x \cdot y = 1,$$

which cannot be transferred to our ${}^*\mathbb{R}_{\mathrm{f}}$, which is only a ring and not a field.

Now, we can formulate the preservation of quantifiers:

Theorem 22. *Let X, $Y \subseteq \mathbb{R}$ and $C \subseteq X \times Y$ be a relation continuous in the domain, then*

$${}^*[p_X(C)]_{\mathrm{f}} = p_{{}^*X_{\mathrm{f}}}({}^*C_{\mathrm{f}}).$$

That is

$${}^*\{x \in X \,|\, \exists y \in Y : \ C(x, y)\}_{\mathrm{f}} = \{x \in {}^*X_{\mathrm{f}} \,|\, \exists y \in {}^*Y_{\mathrm{f}} : \ {}^*C_{\mathrm{f}}(x, y)\}.$$

As a consequence we also have

$$^*\{x \in X \mid \forall y \in Y : \ C(x,y)\}_{\mathrm{f}} = \{x \in {}^*X_{\mathrm{f}} \mid \forall y \in {}^*Y_{\mathrm{f}} : \ {}^*C_{\mathrm{f}}(x,y)\} .$$

Proof. If $[u] \in {}^*[p_X(C)]_{\mathrm{f}}$, then $u_n \in_n p_X(C)$, i.e.

$$\forall^{\mathrm{d}} n : \ u_n \in X \ , \ \exists y \in Y : \ C(u_n, y),$$

that is the set of $n \in \mathbb{N}$ satisfying this relation is dominant. This implies that $u_n \in_n X$ and hence $[u] \in {}^*X_{\mathrm{f}}$ and $u_n \in_n \mathrm{dom}(C)$, i.e. $[u] \in {}^*[\mathrm{dom}(C)]_{\mathrm{f}}$. Our relation C is continuous, so that $[u] \in \mathrm{dom}({}^*C_{\mathrm{f}})$, i.e.

$$\exists \beta \in {}^*Y_{\mathrm{f}} : \ {}^*C_{\mathrm{f}}([u], \beta),$$

which can also be written as

$$[u] \in p_{{}^*X_{\mathrm{f}}}({}^*C_{\mathrm{f}}).$$

To prove the opposite inclusion it suffices to reverse this deduction and use Theorem 19 instead of the Definition 20 of continuous relation. $\square$

Example 23. Let us apply our transfer theorems to a sentence of the form

$$\forall a \in A \, \exists b \in B : \ C(a,b) \tag{5.9}$$

showing that it is equivalent to

$$\forall a \in {}^*A_{\mathrm{f}} \, \exists b \in {}^*B_{\mathrm{f}} : \ {}^*C_{\mathrm{f}}(a,b), \tag{5.10}$$

where $C \subseteq A \times B$ is a binary continuous relation. Assume (5.9) and $a \in {}^*A_{\mathrm{f}}$. From (5.9) we have that

$$A \subseteq \{a \in \mathbb{R} \mid \exists b \in B : \ C(a,b)\} ,$$

Therefore, by Theorem 15 and Theorem 22, we have

$$^*A_{\mathrm{f}} \subseteq \{a \in {}^*\mathbb{R}_{\mathrm{f}} \mid \exists b \in {}^*B_{\mathrm{f}} : \ {}^*C_{\mathrm{f}}(a,b)\}$$

and we obtain the existence of a $b \in {}^*B_{\mathrm{f}}$ such that ${}^*C_{\mathrm{f}}(a,b)$. To prove the opposite implication, it suffices to reverse this deduction.

The ring ${}^*\mathbb{R}_{\mathrm{f}}$ and the field ${}^*\mathbb{R}_{\mathrm{frac}}$ can be used to reformulate proficiently several parts of the calculus. For example, a continuous function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}$ if there exists $m \in \mathbb{R}$ such that for every non zero infinitesimal $h \in {}^*\mathbb{R}_{\mathrm{f}}$

$$\exists \sigma \in {}^*\mathbb{R}_{\mathrm{f}} : \ f(x+h) = f(x) + h \cdot m + h \cdot \sigma \quad , \quad \sigma \approx 0,$$

that is if $f(x+h)$ is equal to the tangent line $y = f(x) + h \cdot m$ up to an infinitesimal of order greater than h, i.e. of the form $h \cdot \sigma$, with $\sigma \approx 0$. Because ${}^*\mathbb{R}_{\mathrm{f}}$ is an integral domain, taking a non zero infinitesimal h,

we can easily prove that such $m \in \mathbb{R}$ is unique. Working in ${}^*\mathbb{R}_{\text{frac}}$, we have that such m is given by

$$m = \operatorname{st}\left(\frac{f(x+h)-f(x)}{h}\right).$$

Of course, this real number will be denoted by $f'(x)$, so that we have that it is infinitely close (or, in Fermat's terminology, *adequal*; see [35, p. 28]) to the corresponding infinitesimal ratio

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}.$$

Reformulation is only the most trivial possibility offered by a continuum with infinitesimals, because our geometrical and physical intuition is now strongly supported by a corresponding rigorous mathematical formalism.

6. A possible approach with nilpotent infinitesimals

There is another approach of refining Cantor equivalence relation on real Cauchy sequences. This approach avoiding ultrafilters. The idea is to compare two sequences u, $v \in \mathcal{C}$ with a basic infinitesimal, e.g. $\left(\frac{1}{n}\right)_n$. We therefore set by definition

$$u \sim v \quad \iff \quad \lim_{n\to+\infty} n \cdot (u_n - v_n) = 0. \tag{6.1}$$

In other words, using Landau's little-oh notation, the two Cauchy sequences are to be equivalent if

$$u_n = v_n + o\left(\tfrac{1}{n}\right) \quad \text{for } n \to +\infty.$$

As in the previous part of the article, we will denote the equivalence class of a sequence u simply by $[u]$. The relation defined in (6.1) is stronger than the usual Cauchy relation:

$$u \sim v \quad \Rightarrow \quad \exists \lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n =: \operatorname{st}([u]) \in \mathbb{R}.$$

It is also strictly stronger, because, e.g., the equivalence class $\left[\left(\frac{1}{n^p}\right)_n\right]$, with $0 < p \leq 1$, is a nonzero infinitesimal. For example, the infinitesimal $\left[\left(\frac{1}{n}\right)_n\right]$ is not zero, but we can think of it as being so small that its square is zero: $\left[\left(\frac{1}{n^2}\right)_n\right] = [0]$. With respect to pointwise operations, we thus obtain a ring rather than a field. A ring with nilpotent elements may seem unwieldy; however, this was surely not the case for geometers like S. Lie, E. Cartan, A. Grothendieck, or for physicists like P.A.M.

Dirac or A. Einstein (see, e.g., references in [8]). The latter used to write formulas, if $v/c \ll 1$, like

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{v^2}{2c^2}$$

containing an equality sign rather than an approximate equality sign. More generally, in [7] A. Einstein wrote

$$f(x, t+\tau) = f(x,t) + \tau \cdot \frac{\partial f}{\partial t}(x,t) \tag{6.2}$$

justifying it with the words "*since τ is very small*". Let us note that if we apply (6.2) to the function $f(x,t) = t^2$ at $t = 0$, we obtain $\tau^2 = 0 + \tau \cdot 0 = 0$ and therefore we necessarily obtain that our ring of scalars contains nilsquare elements. Of course, it is not easy to state that physicists like A. Einstein or P.A.M. Dirac were consciously working with this kind of scalars; indeed, their work, even if it is sometimes found to be lacking from the formal/syntactical point of view, it is always strongly supported by a strong bridge with the physical meaning of the relationships being discovered.

A difficult point in working with a ring having nilpotent elements is the concrete management of powers of nilpotent elements, like

$$h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}.$$

Let us note that this kind of product appears naturally in several variable Taylor formulae. Is such a product zero or not? Are we able to decide effectively whether it is zero starting from the properties of the infinitesimals h_j and the exponents i_j? To be able to give an affirmative answer to this, and several other questions, we restrict this construction to a particular subclass of Cauchy sequences, as follows.

Definition 24. We say that u is a *little-oh polynomial*, and we write $u \in \mathbb{R}_o\left[\frac{1}{n}\right]$ if and only if we can write

$$u_n = r + \sum_{i=1}^{k} \alpha_i \cdot \frac{1}{n^{a_i}} + o\left(\frac{1}{n}\right) \quad \text{as} \quad n \to +\infty, \tag{6.3}$$

for suitable $k \in \mathbb{N}$, $r, \alpha_1, \ldots, \alpha_k \in \mathbb{R}$, $a_1, \ldots, a_k \in \mathbb{R}_{\geq 0}$.

Therefore, $\mathbb{R}_o\left[\frac{1}{n}\right] \subset \mathcal{C}$ and our previous example $\left[\left(\frac{1}{n^p}\right)_n\right]$ is generated by a little-oh polynomial. Little-oh polynomials are closed with respect to pointwise ring operations, and the corresponding quotient ring

$${}^{\bullet}\mathbb{R} := \mathbb{R}_o\left[\frac{1}{n}\right] / \sim$$

is called ring of *Fermat reals*. The name is motivated essentially by two reasons: in the ring of Fermat reals, it is possible to formalize the informal method used by A. Fermat to find maxima and minima, see [11]; all the theory of Fermat reals and Fermat extensions has been constructed trying always to have a strong bridge between formal properties and informal geometrical interpretation: we think that this has been one of the leading methods used by A. Fermat in his work. For all the proofs of this section, we refer to [10, 8].

Exactly as in the previous section about the hyperreals, we have that the ring of Fermat reals ${}^\bullet\mathbb{R}$ is still Cauchy complete with respect to the pseudo-metric

$$d\,(x,y) := |\mathrm{st}(x) - \mathrm{st}(y)| \in \mathbb{R} \quad \forall x, y \in {}^\bullet\mathbb{R}.$$

Once again, the ring ${}^\bullet\mathbb{R}$ is not Dedekind complete.

It is not hard to prove that all the numbers k, r, α_i, a_i appearing in (6.3) are uniquely determined if we impose upon them the constraints

$$0 < a_1 \le a_2 \le \cdots \le a_k \le 1 \tag{6.4}$$

$$\alpha_i \neq 0 \quad \forall i = 1, \ldots, k. \tag{6.5}$$

We can therefore introduce the following notation.

Definition 25. If $x := [u] \in {}^\bullet\mathbb{R}$ and k, r, α_i, a_i are the unique real numbers appearing in (6.3) and satisfying (6.4) and (6.5), then we set ${}^\circ x := \mathrm{st}(x) := r$, ${}^\circ x_i := \alpha_i$, $\omega(x) := \frac{1}{a_1}$, $\omega_i(x) := \frac{1}{a_i}$, $N_x := k$. Moreover, we set

$$\mathrm{d}t_a := \left[\left(\sqrt[a]{\frac{1}{n}} \right)_n \right] \in {}^\bullet\mathbb{R} \quad \forall a \in \mathbb{R}_{>0}$$

and, more simply, $\mathrm{d}t := \mathrm{d}t_1$. Using these notations, we can write any Fermat real as

$$x = {}^\circ x + \sum_{i=1}^{N_x} {}^\circ x_i \cdot \mathrm{d}t_{\omega_i(x)} \tag{6.6}$$

where the equality sign has to be meant in ${}^\bullet\mathbb{R}$. The numbers ${}^\circ x_i$ are called the *standard parts* of x and the numbers $\omega_i(x)$ the *orders* of x (for $i = 1$ we will simply use the names *standard part* and *order* for ${}^\circ x$ and $\omega(x)$). The unique writing (6.6) is called the *decomposition of* x.

Let us note the following properties of the infinitesimals of the form $\mathrm{d}t_a$:

$$\mathrm{d}t_a \cdot \mathrm{d}t_b = \mathrm{d}t_{\frac{ab}{a+b}}$$

$$(\,\mathrm{d}t_a)^p = \mathrm{d}t_{\frac{a}{p}} \quad \forall p \in \mathbb{R}_{\ge 1}$$

$$\mathrm{d}t_a = 0 \quad \forall a \in \mathbb{R}_{<1}.$$

A first justification to the name "order" is given by the following

Theorem 26. *If* $x \in {}^\bullet\mathbb{R}$ *and* $k \in \mathbb{N}_{>1}$*, then* $x^k = 0$ *in* ${}^\bullet\mathbb{R}$ *if and only if* ${}^\circ x = 0$ *and* $\omega(x) < k$*.*

This motivates also the definition of the following ideal of infinitesimals:

Definition 27. If $a \in \mathbb{R} \cup \{\infty\}$, then

$$D_a := \{x \in {}^\bullet\mathbb{R} \mid {}^\circ x = 0\ ,\ \omega(x) < a + 1\}\,.$$

These ideals are naturally tied with the infinitesimal Taylor formula (i.e. without any rest because of the use of nilpotent infinitesimal increments), as one can guess from the property

$$a \in \mathbb{N} \quad \Rightarrow \quad D_a = \left\{x \in {}^\bullet\mathbb{R} \mid x^{a+1} = 0\right\}.$$

Products of powers of nilpotent infinitesimals can be effectively decided using the following result

Theorem 28. *Let* $h_1, \dots, h_n \in D_\infty \setminus \{0\}$ *and* $i_1, \dots, i_n \in \mathbb{N}$*, then*

(1) $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} = 0 \iff \sum_{k=1}^n \frac{i_n}{\omega(h_n)} > 1$
(2) $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} \neq 0 \ \Rightarrow \ \frac{1}{\omega\left(h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}\right)} = \sum_{k=1}^n \frac{i_n}{\omega(h_n)}$.

This result motivates strongly our choice to restrict our construction to little-oh polynomials only.

The reader can naturally ask what would happen in case of a different choice of the basic infinitesimal $\left(\frac{1}{n}\right)_n$ in the Definition (6.1). Really, any other choice of a different infinitesimal $(s_n)_n$ will conduct to an isomorphic ring through the isomorphism

$${}^\circ x + \sum_{i=1}^{N_x} {}^\circ x_i \cdot \mathrm{d}t_{\omega_i(x)} \mapsto \left[\left({}^\circ x + \sum_{i=1}^{N_x} {}^\circ x_i \cdot s_n^{\frac{1}{\omega_i(x)}}\right)_n\right]_\sim$$

This is the only ring isomorphism preserving the basic infinitesimals $\mathrm{d}t_a$ and the standard part function, i.e. such that:

$$f\left(\alpha \cdot \mathrm{d}t_a\right) = \alpha \cdot \left[\left(\sqrt[a]{s_n}\right)_n\right]_\sim$$
$$f({}^\circ x) = {}^\circ f(x).$$

Essentially the same isomorphism applies also to the ring defined in [10], where instead of sequences, the construction is based on real functions of the form $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$.

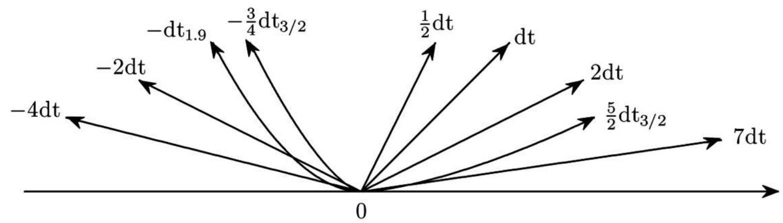

FIGURE 7.1. Some first order infinitesimals

7. ORDER RELATION

It is not hard to define an intuitively meaningful order relation on the ring of Fermat reals

Definition 29. Let x, $y \in {}^\bullet\mathbb{R}$ be Fermat reals, then we say that $x \le y$ iff we can find representatives $[u] = x$ and $[v] = y$ such that

$$\exists N \in \mathbb{N}\, \forall n \ge N : \ u_n \le v_n.$$

For all the proofs of this section, see e.g. [12, 8].

It is not hard to show that this relation is well defined on ${}^\bullet\mathbb{R}$ and that the induced ordered relation is total. This is another strong motivation for the choice of little-oh polynomials in the construction of the ring of Fermat reals. The analogous of Theorem 5 is the following

Theorem 30. *Let $h \in {}^\bullet\mathbb{R}$, then the following are equivalent*

(1) $h \in D_\infty$, *i.e.* ${}^\circ h = 0$, *i.e. h is an infinitesimal*
(2) $\forall n \in \mathbb{N}_{>0} : \ -\frac{1}{n} < h < \frac{1}{n}$

The following theorem permits to decide algorithmically the order relation between two Fermat reals, using only their decompositions

Theorem 31. *Let x,$y \in {}^\bullet\mathbb{R}$. If ${}^\circ x \ne {}^\circ y$, then*

$$x < y \iff {}^\circ x < {}^\circ y.$$

Otherwise, if ${}^\circ x = {}^\circ y$, then

(1) *If $\omega(x) > \omega(y)$, then $x > y$ if and only if ${}^\circ x_1 > 0$.*
(2) *If $\omega(x) = \omega(y)$, then*

$$\begin{aligned} {}^\circ x_1 > {}^\circ y_1 \ &\Rightarrow \ x > y \\ {}^\circ x_1 < {}^\circ y_1 \ &\Rightarrow \ x < y. \end{aligned}$$

For example, $0 < \mathrm{d}t < \mathrm{d}t_2 < \mathrm{d}t_3$, etc. This motivates why we take $\frac{1}{a}$ in the definition of $\mathrm{d}t_a$: in this way the greater is the order a and the greater is the infinitesimal.

The ring ${}^\bullet\mathbb{R}$ can also be represented geometrically.

Definition 32. If $x \in {}^\bullet\mathbb{R}$ and $\delta \in \mathbb{R}_{>0}$, then

$$\mathrm{graph}_\delta(x) := \left\{ ({}^\circ x + \sum_{i=1}^{N_x} {}^\circ x_i \cdot t^{1/\omega_i(x)}, t) \mid 0 \le t < \delta \right\} \tag{7.1}$$

E.g. $\mathrm{graph}_\delta(\,\mathrm{d}t_2) = \{(\sqrt{t}, t) | 0 \le t < \delta\}$. Note that the values of the function are placed in the abscissa position, so that the correct representation of $\mathrm{graph}_\delta(x)$ is given by the figure 7.1. This inversion of abscissa and ordinate in the $\mathrm{graph}_\delta(x)$ permits to represent this graph as a line tangent to the classical straight line $\mathbb{R}$ and hence to have a better graphical picture. Finally, note that if $x \in \mathbb{R}$ is a standard real, then $N_x = 0$ and the $\mathrm{graph}_\delta(x)$ is a vertical line passing through ${}^\circ x = x$, i.e. they are "ticks on axis".

The following theorem introduces the geometric representation of the ring of Fermat reals.

Theorem 33. *If $\delta \in \mathbb{R}_{>0}$, then the function*

$$x \in {}^\bullet\mathbb{R} \mapsto \mathrm{graph}_\delta(x) \subset \mathbb{R}^2$$

is injective. Moreover if x, $y \in {}^\bullet\mathbb{R}$, then we can find $\delta \in \mathbb{R}_{>0}$ (depending on x and y) such that

$$x < y$$

if and only if

$$\forall p, q, t : \ (p, t) \in \mathrm{graph}_\delta(x) \ , \ (q, t) \in \mathrm{graph}_\delta(y) \ \Rightarrow \ p < q \tag{7.2}$$

that is if a point (p, t) on $\mathrm{graph}_\delta(x)$ comes before (with respect to the order on the x-axis) the corresponding point (q, t) on $\mathrm{graph}_\delta(y)$.

8. Infinitesimal Taylor formula and computer implementation

What kind of functions $f : \mathbb{R} \to \mathbb{R}$ can be extended on ${}^\bullet\mathbb{R}$? The idea for the definition of extension is natural ${}^\bullet f([u]) := [f \circ u]$ so that we have to chose f so that:

(1) If u is a little-oh polynomial, then also $f \circ u$ is a little-oh polynomial.
(2) If $[u] = [v]$, then also $[f \circ u] = [f \circ v]$.

The second condition is surely satisfied if we take f locally Lipschitz, but the first one holds if f is smooth.

Definition 34. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function, then

$${}^\bullet f([u_1], \dots, [u_d]) := [f \circ u_1, \dots, f \circ u_d] \quad \forall [u_1], \dots, [u_d] \in {}^\bullet\mathbb{R}.$$

Therefore, the ring of Fermat reals seems potentially useful e.g. for smooth differential geometry (see e.g. chapter 13 of [8]) or in some part of physics (see e.g. [9]), where one can suppose to deal only with smooth functions.

In several applications, the following infinitesimal Taylor formulae permit to formalize perfectly the informal results frequently appearing in physics.

Theorem 35. *Let $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ a smooth function, then*

$$\exists!\, m \in \mathbb{R}\ \forall h \in D_1 :\ f(x+h) = f(x) + h \cdot m. \tag{8.1}$$

In this case we have $m = f'(x)$, where $f'(x)$ is the usual derivative of f at x.

Theorem 36. *Let $x \in \mathbb{R}^d$, $n \in \mathbb{N}_{>0}$ and $f : \mathbb{R}^d \to \mathbb{R}$ a smooth function, then*

$$\forall h \in D_n^d :\ f(x+h) = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \le n}} \frac{h^j}{j!} \cdot \frac{\partial^{|j|} f}{\partial x^j}(x).$$

Note that $m = f'(x) \in \mathbb{R}$ in Theorem 35, i.e. the slope is a standard real number, and that we can use this formula with standard real numbers x only, and not with a generic $x \in {}^\bullet\mathbb{R}$, but it is possible to remove these limitations (see [11, 9, 8]).

The definition of the ring of Fermat reals is highly constructive. Therefore, using object oriented programming, it is not hard to write a computer code corresponding to ${}^\bullet\mathbb{R}$. We (see also [13]) realized a first version of this software using Matlab R2010b.

The constructor of a Fermat real is `x=FermatReal(s,w,r)`, where `s` is the $n+1$ double vector of standard parts (`s(1)` is the standard part ${}^\circ x$) and `w` is the double vector of orders (`w(1)` is the order $\omega(x)$ if $x \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$, otherwise `w=[]` is the empty vector). The last input `r` is a logical variable and assumes value `true` if we want that the display of the number `x` is realized using the Matlab `rats` function for both its standard parts and orders. In this way, the number will be displayed using continued fraction approximations and therefore, in several cases, the calculations will be exact. These inputs are the basic methods of every Fermat real, and can be accessed using the `subsref`, and `subsasgn`, notations `x.stdParts`, `x.orders`, `x.rats`. The function `w=orders(x)` gives exactly the double vector `x.orders` if $x \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$ and `0` otherwise.

The function `dt(a)`, where `a` is a double, construct the Fermat real $\mathrm{d}t_a$. Because we have overloaded all the algebraic operations, like `x+y`, `x*y`, `x-y`, `-x`, `x==y`, `x~=y`, `x<y`, `x<=y`, `x^y`, we can define a Fermat real e.g. using an expression of the form `x=2+3*dt(2)-1/3*dt(1)`, which corresponds to `x=FermatReal([2 3 -1/3],[2 1],true)`.

We have also realized the function `y=decomposition(x)`, which gives the decomposition of the Fermat real `x`, and the functions `abs(x)`, `log(x)`, `exp(x)`, `isreal(x)`, `isinfinitesimal(x)`, `isinvertible(x)`.

The function `plot(t,x)` shows the curve (7.1) at the given input `t` of double.

The ratio `x/y` has been implemented if `y` is invertible. Finally, the function `y=ext(f,x)`, corresponds to ${}^\bullet f(x)$ and has been realized using the evaluation of the symbolic Taylor formula of the inline function `f`.

Using these tools, we can easily find, e.g., that

$$\frac{\sin(\,\mathrm{d}t_3 + 2\,\mathrm{d}t_2)}{\cos(-\,\mathrm{d}t_4 - 4\,\mathrm{d}t)} = \mathrm{d}t_3 + 2\,\mathrm{d}t_2 - \frac{1}{2}\,\mathrm{d}t_{\frac{6}{5}} + \frac{5}{6}\,\mathrm{d}t.$$

This corresponds to the following Matlab code:

```
>> x=dt(3)+2*dt(2)
  x =
  dt_3 + 2*dt_2
>> y=-dt(4)-4*dt(1)
  y =
  -dt_4 - 4*dt
>> g=inline('cos(y)')
  g =
  Inline function:  g(y) = cos(y)
  >> f=inline('sin(x)')
  f =
  Inline function:  f(x) = sin(x)
>> decomposition(ext(f,x)/ext(g,y))
  ans =
  dt_3 + 2*dt_2 + 1/2*dt_6/5 + 5/6*dt
```

The Matlab source code is freely available under open-source licence, and can be requested to the authors of the present article.

9. Leibniz's law of continuity in ${}^\bullet\mathbb{R}$

Is a suitable form of the Leibniz's law of continuity provable in the ring of Fermat reals? The first version is the transfer for equality and inequality, that can be proved proceeding like in Theorem 9.

Theorem 37. *Let $f,\, g : \mathbb{R}^d \to \mathbb{R}$ be smooth functions, then it results*

$$\forall x_1, \ldots, x_d \in \mathbb{R} : \;\; f(x_1, \ldots, x_d) = g(x_1, \ldots, x_d)$$

if and only if

$$\forall x_1, \ldots, x_d \in {}^\bullet\mathbb{R} : \;\; {}^\bullet f(x_1, \ldots, x_d) = {}^\bullet g(x_1, \ldots, x_d).$$

Analogously, we can formulate the transfer of inequalities of the form $f(x_1, \ldots, x_d) < g(x_1, \ldots, x_d)$.

Now, we can proceed as for ${}^*\mathbb{R}_{\mathrm{f}}$. We firstly define the extension ${}^\bullet U$ of a generic subset $U \subseteq \mathbb{R}$.

Definition 38. Define the set of little-oh polynomials $U_o\left[\frac{1}{n}\right]$ as in Definition 24 but taking sequences $u : \mathbb{N} \to U$ with values in U and such that ${}^\circ[u] := \lim_{n\to+\infty} u_n \in U$. For $u,\, v \in U_o\left[\frac{1}{n}\right]$ define $u \sim v$ for $u_n = v_n + o\left(\frac{1}{n}\right)$ as $n \to +\infty$ and ${}^\bullet U := U_o\left[\frac{1}{n}\right] / \sim$.

If $i : U \hookrightarrow \mathbb{R}$ is the inclusion map, it is easy to prove that its Fermat extension ${}^\bullet i : {}^\bullet U \to {}^\bullet\mathbb{R}$ is injective. We will always identify ${}^\bullet U$ with ${}^\bullet i({}^\bullet U)$, so we simply write ${}^\bullet U \subseteq {}^\bullet\mathbb{R}$. According to this identification, if U is open in $\mathbb{R}$, we can also prove that

$${}^\bullet U = \{x \in {}^\bullet\mathbb{R} \mid {}^\circ x \in U\}. \tag{9.1}$$

Because of our Theorem 12 we must expect that our extension operator ${}^\bullet(-)$ doesn't preserve all the operators of propositional logic like "and", "or" and "not". To guess what kind of preservation properties hold for this operator we say that the theory of Fermat reals is strongly inspired by synthetic differential geometry (SDG; see, e.g., [26, 31, 1]). SDG is the most beautiful and powerful theory of nilpotent infinitesimals with important applycations to differential geometry of both finite and infinite dimensional spaces. Its models require a certain knowledge of Topos theory, because a model in classical logic is not possible. Indeed, the internal logic of its topos models is necessarily intuitionistic. Fermat reals have several analogies with SDG even if, at the end it is a completely different theory. For example, in ${}^\bullet\mathbb{R}$ the product of any two first order infinitesimals is always zero, whereas in SDG this is not the case. On the other hand, the intuitive interpretation of Fermat reals is stronger and there is full compatibility with classical logic.

This background explain why we will show that our extension operator preserves intuitionistic logical operations. Even if the theory of Fermat reals can be freely studied in classical logic[8], the "most natural logic" of smooth spaces and smooth functions remains the intuitionistic one. We simply recall here that the intuitionistic Topos models of SDG

[8]More generally, without requiring a background in formal logic.

show formally that L.E.J. Brouwer's idea of the impossibility to define a non smooth functions without using the law of excluded middle or the axiom of choice is correct.

Because we need to talk of open sets both in $\mathbb{R}$ and in ${}^\bullet\mathbb{R}$ we have to introduce the following

Definition 39. We always think on ${}^\bullet\mathbb{R}$ the so-called *Fermat topology*, i.e. the topology generated by subsets of the form ${}^\bullet U \subseteq {}^\bullet\mathbb{R}$ for U open in $\mathbb{R}$.

Theorem 40. *Let A, B be open sets of $\mathbb{R}$, then the following preservation properties hold*

(1) ${}^\bullet(A \cup B) = {}^\bullet A \cup {}^\bullet B$
(2) ${}^\bullet(A \cap B) = {}^\bullet A \cap {}^\bullet B$
(3) ${}^\bullet\mathrm{int}(A \setminus B) = \mathrm{int}({}^\bullet A \setminus {}^\bullet B)$
(4) $A \subseteq B$ *if and only if* ${}^\bullet A \subseteq {}^\bullet B$
(5) ${}^\bullet\emptyset = \emptyset$
(6) ${}^\bullet A = {}^\bullet B$ *if and only if* $A = B$

Proof. We will use frequently the characterization (9.1). To prove (1) we have that $x \in {}^\bullet(A \cup B)$ iff $x \in {}^\bullet\mathbb{R}$ and ${}^\circ x \in A \cup B$, i.e. iff ${}^\circ x \in A$ or ${}^\circ x \in B$ and, using again (9.1), this happens iff $x \in {}^\bullet A$ or $x \in {}^\bullet B$. Analogously, we can prove (2). We firstly prove (4). If $A \subseteq B$ and $x \in {}^\bullet A$, then ${}^\circ x \in A$ and hence also ${}^\circ x \in B$ and $x \in {}^\bullet B$. Viceversa if ${}^\bullet A \subseteq {}^\bullet B$ and $a \in A$, then ${}^\circ a = a$ so that $a \in {}^\bullet B$, that is ${}^\circ a = a \in B$. To prove (3) we have that $x \in {}^\bullet\mathrm{int}(A \setminus B)$ iff ${}^\circ x \in \mathrm{int}(A \setminus B)$, i.e. iff $({}^\circ x - \delta, {}^\circ x - \delta) \subseteq A \setminus B$ for some $\delta \in \mathbb{R}_{>0}$. From (4) we have ${}^\bullet({}^\circ x - \delta, {}^\circ x - \delta) \subseteq {}^\bullet A$ and $x \in {}^\bullet({}^\circ x - \delta, {}^\circ x - \delta)$. Finally, a generic $y \in {}^\bullet({}^\circ x - \delta, {}^\circ x - \delta)$ cannot belong to ${}^\bullet B$ because, otherwise, ${}^\circ y \in ({}^\circ x - \delta, {}^\circ x - \delta) \cap B$ which is impossible. Therefore, x is internal to ${}^\bullet A \setminus {}^\bullet B$ with respect to the Fermat topology. The proofs of (5) and (6) are direct or follow directly from (4). $\square$

Example 41. Using the previous theorem, we can prove the transfer of the analogous of (5.5), but where we need now to suppose that A, B, C are open subsets of $\mathbb{R}$. Therefore, we have

$$\forall x \in \mathbb{R} : \ A(x) \ \Rightarrow \ [B(x) \text{ and } (C(x) \ \Rightarrow \ D(x))]$$

if and only if

$$\forall x \in {}^\bullet\mathbb{R} : \ {}^\bullet A(x) \ \Rightarrow \ [{}^\bullet B(x) \text{ and } ({}^\bullet C(x) \ \Rightarrow \ {}^\bullet D(x))] \, .$$

Once again, we don't strictly need a background of intuitionistic logic to understand that the preservation of quantifier for the Fermat extension ${}^\bullet(-)$ must be formulated in the following way

Theorem 42. *Let A, B be open subsets of $\mathbb{R}$, and C be open in $A\times B$. Let $p:(a,b)\in A\times B\mapsto a\in A$ be the projection on the first component. Define*

$$\begin{aligned}
{}^\bullet C &:= \{(\alpha,\beta)\,|\,({}^\circ\alpha,{}^\circ\beta)\in C\}\\
\exists_p(C) &:= p(C)\\
\forall_p(C) &:= \mathrm{int}\left(A\setminus\exists_p\left(\mathrm{int}\left((A\times B)\setminus C\right)\right)\right).
\end{aligned}$$

Then

$$\begin{aligned}
{}^\bullet\left[\exists_p(C)\right] &= \exists_{{}^\bullet p}({}^\bullet C)\\
{}^\bullet\left[\forall_p(C)\right] &= \forall_{{}^\bullet p}({}^\bullet C).
\end{aligned}$$

That is

$$\begin{aligned}
{}^\bullet\left\{a\in A\,|\,\exists b\in B:\ C(a,b)\right\} &= \left\{a\in{}^\bullet A\,|\,\exists b\in{}^\bullet B:\ {}^\bullet C(a,b)\right\}\\
{}^\bullet\left\{a\in A\,|\,\forall b\in B:\ C(a,b)\right\} &= \left\{a\in{}^\bullet A\,|\,\forall b\in{}^\bullet B:\ {}^\bullet C(a,b)\right\}.
\end{aligned}$$

Proof. The preservation of the universal quantifier follows from that of the existential quantifier and from property (3) of 40, so that we only have to prove ${}^\bullet\left[p(C)\right]={}^\bullet p({}^\bullet C)$. Consider that the projection is an open map, so that $p(C)$ is open because C is open in $A\times B$. Therefore $x\in{}^\bullet\left[p(C)\right]$ iff ${}^\circ x\in p(C)$, and this holds iff we can find $(a,b)\in C$ such that ${}^\circ x=p(a,b)=a\in A$. Therefore, ${}^\bullet p(x,b)=\left[\left(p(x_n,b)\right)_n\right]=[(x_n)_n]=x$ and $(x,b)\in{}^\bullet C$ because $({}^\circ x,b)=(a,b)\in C$. This proves that ${}^\bullet\left[p(C)\right]\subseteq{}^\bullet p({}^\bullet C)$. Vice versa, if $x\in{}^\bullet p({}^\bullet C)$, then we can find $(\alpha,\beta)\in{}^\bullet C$ such that $x={}^\bullet p(\alpha,\beta)=\alpha$. Therefore, $({}^\circ\alpha,{}^\circ\beta)\in C$ and $p({}^\circ\alpha,{}^\circ\beta)={}^\circ\alpha={}^\circ x$. This means that ${}^\circ x\in p(C)$, which is open and hence $x\in{}^\bullet p(C)$. ☐

Example 43. Using the previous theorem, we can prove the transfer of the analogous of Example 23, but where we need now to suppose that A, B are open subsets of $\mathbb{R}$ and C is open in $A\times B$. Therefore, we have

$$\forall a\in A\,\exists b\in B:\ C(a,b)$$

if and only if

$$\forall a\in{}^\bullet A\,\exists b\in{}^\bullet B:\ {}^\bullet C(a,b).$$

The theory of Fermat reals can be greatly developed: any smooth manifold can be extended with similar infinitely closed points and the extension functor ${}^\bullet(-)$ has wonderful preservation properties that generalize what we have just seen on the (intuitionistic) Leibniz's law of continuity in ${}^\bullet\mathbb{R}$. Potential useful applycations are in the differential geometry of spaces of functions, like the space of all the smooth functions between two manifolds.

10. Conclusion

We started with the idea of refining the equivalence relation among real Cauchy sequences so as to obtain a new infinitesimal-enriched continuum. We have developed this idea in two directions. The first direction takes one toward the hyperreals, and we tried to motivate the choices one must make to arrive at a powerful theory. On the other hand, we saw that the intuitive interpretation of such choices is sometimes lacking. The second idea is intuitively clearer but surely formally less powerful. The two ideas serve different scopes because they deal with different kinds of infinitesimals: invertible and nilpotent.

References

[1] J.L. Bell, *A Primer of Infinitesimal Analysis*, Cambridge University Press, 1998.

[2] V. Benci and M. Di Nasso. A purely algebraic characterization of the hyperreal numbers. *Proceedings of the American Mathematical Society*, 133(9):2501–05, 2005.

[3] Borovik, A.; Katz, M.: Who gave you the Cauchy–Weierstrass tale? The dual history of rigorous calculus. *Foundations of Science*, 2011, see http://dx.doi.org/10.1007/s10699-011-9235-x and http://arxiv.org/abs/1108.2885

[4] Cauchy, A. L.: *Cours d'Analyse de L'Ecole Royale Polytechnique. Première Partie. Analyse algébrique* (Paris: Imprimérie Royale, 1821).

[5] N. Cutland, C. Kessler, E. Kopp, and D. Ross. On Cauchy's notion of infinitesimal. *British J. Philos. Sci.*, 39(3):375–378, 1988.

[6] P. Ehrlich. The rise of non-archimedean mathematics and the roots of a misconception I: The emergence of non-archimedean systems of magnitudes. *Archive for History of Exact Sciences*, 60(1):1–121, 2006.

[7] A. Einstein. *Investigations on the Theory of the Brownian Movement.* Dover, 1926.

[8] P. Giordano. Fermat reals: Nilpotent infinitesimals and infinite dimensional spaces. arXiv:0907.1872, July 2009.

[9] P. Giordano. Infinitesimals without logic. *Russian Journal of Mathematical Physics*, 17(2):159–191, 2010.

[10] P. Giordano. The ring of fermat reals. *Advances in Mathematics*, 225(4):2050–2075, 2010.

[11] P. Giordano. Fermat-Reyes method in the ring of Fermat reals. To appear in *Advances in Mathematics*, 2011.

[12] P. Giordano. Order relation and geometrical representation of Fermat reals. Submitted to *Annali della SNS*, 2011.

[13] P. Giordano and M. Kunzinger. Topological and algebraic structures on the ring of Fermat reals. Submitted to *Israel Journal of Mathematics*, 2011.

[14] Goldblatt, R.: Lectures on the hyperreals. An introduction to nonstandard analysis. Graduate Texts in Mathematics, 188. Springer-Verlag, New York, 1998.

[15] C.W. Henson. Foundations of nonstandard analysis. A gentle introduction to nonstandard extension. In L.O. Arkeryd, N.J. Cutland, and C.W. Henson, editors, *Nonstandard analysis: theory and applications (Edinburgh, 1996)*, pages 1–49, Dordrecht, 1997. NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., vol. 493, Kluwer Acad. Publ.

[16] E. Hewitt. Rings of real-valued continuous functions. I. *Trans. Amer. Math. Soc.*, 64:45–99, 1948.

[17] Hrbacek, K.; Lessmann, O.; O'Donovan, R.: Analysis with ultrasmall numbers. *Amer. Math. Monthly* **117** (2010), no. 9, 801–816.

[18] Katz, K.; Katz, M.: Zooming in on infinitesimal $1 - .9..$ in a post-triumvirate era. *Educational Studies in Mathematics* **74** (2010), no. 3, 259-273. See arXiv:1003.1501.

[19] Katz, K.; Katz, M.: When is .999... less than 1? *The Montana Mathematics Enthusiast* **7** (2010), No. 1, 3–30.

[20] Katz, K.; Katz, M.: A Burgessian critique of nominalistic tendencies in contemporary mathematics and its historiography. *Foundations of Science* (2011), see http://dx.doi.org/10.1007/s10699-011-9223-1
and http://arxiv.org/abs/1104.0375

[21] Katz, K.; Katz, M.: Cauchy's continuum. *Perspectives on Science* **19** (2011), no. 4, 426-452. See http://www.mitpressjournals.org/toc/posc/19/4
and http://arxiv.org/abs/1108.4201

[22] Katz, K.; Katz, M.: Stevin numbers and reality, *Foundations of Science*, 2011, see http://dx.doi.org/10.1007/s10699-011-9228-9
and http://arxiv.org/abs/1107.3688

[23] Katz, M.; Tall, D.: The tension between intuitive infinitesimals and formal mathematical analysis. Chapter in book Bharath Sriraman, Editor.
http://www.infoagepub.com/products/Crossroads-in-the-History-of-Mathematics

[24] J.H. Keisler. *Elementary Calculus: An Approach Using Infinitesimals*. Prindle Weber & Schmidt, 1976.

[25] E. Knobloch. Leibniz's rigorous foundation of infinitesimal geometry by means of Riemannian sums. Foundations of the formal sciences, 1 (Berlin, 1999). *Synthese* **133** (2002), no. 1-2, 59–73.

[26] A. Kock. *Synthetic Differential Geometry*, volume 51 of London Math. Soc. Lect. Note Series. Cambridge Univ. Press, 1981.

[27] D. Laugwitz. Leibniz' principle and omega calculus. [A] Le labyrinthe du continu, Colloq., Cerisy-la-Salle/Fr. 1990, 144-154 (1992).

[28] J. Łoś. *Quelques remarques, thormes et problmes sur les classes dfinissables d'algbres*, volume Mathematical interpretation of formal systems, pages 98–113. North-Holland Publishing Co., Amsterdam, 1955.

[29] W.A.J. Luxemburg. Reduced powers of the real number system and equivalents of the Hahn-Banach extension theorem. 1969 Applications of Model Theory to Algebra, Analysis, and Probability (Internat. Sympos., Pasadena, Calif., 1967) pp. 123–137. Holt, Rinehart and Winston, New York.

[30] W.A.J. Luxemburg. *Non-Standard Analysis: Lectures on A. Robinson's Theory of Infinitesimals and Infinitely Large Numbers*. California Institute of Technology, Pasadena, California, 1973.

[31] I. Moerdijk, G.E. Reyes, *Models for Smooth Infinitesimal Analysis*, Springer, Berlin, 1991.

[32] Nelson, E.: Internal set theory: a new approach to nonstandard analysis. *Bull. Amer. Math. Soc.* **83** (1977), no. 6, 1165–1198.
[33] A. Robinson. *Non-standard analysis*. North-Holland Publishing Co., Amsterdam, 1966.
[34] A. Tarski. Une contribution la thorie de la mesure, *Fund. Math.* **15** (1930), 42-50.
[35] A. Weil. *Number theory. An approach through history. From Hammurapi to Legendre*. Birkhuser Boston, Inc., Boston, MA, 1984.

Department of Mathematics, University of Vienna, Austria.
E-mail address: paolo.giordano@univie.ac.at

Department of Mathematics, Bar Ilan University, Ramat Gan, 52900 Israel.
E-mail address: katzmik@macs.biu.ac.il